PHYS 798C Spring 2022
Lecture 17 Summary

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I. SUPERCONDUCTORS IN A MAGNETIC FIELD - H_{c2} AND VORTICES

We explore how superconductivity nucleates in a strong magnetic field.

A. Linearized GL Equation

We consider the onset of superconductivity for $T < T_c$ in a strong magnetic field as the field is reduced. We look for the onset of superconductivity under this demanding situation in which $|\psi|^2 << |\psi_\infty|^2$. In this case we can neglect the $\psi|\psi|^2$ term in the GL equation to linearize it:
$\alpha\psi + \frac{1}{2m^*}\left(\frac{\hbar}{i}\vec{\nabla} - e^*\vec{A}\right)^2\psi = 0.$
We can also write this linearized GL equation as
$\left(\frac{1}{i}\vec{\nabla} - \frac{2\pi}{\Phi_0}\vec{A}\right)^2\psi = \frac{1}{\xi_{GL}^2}\psi.$
The expression for the current density $\vec{J} = \frac{e^*}{m^*}|\psi|^2\left(\hbar\vec{\nabla}\phi - e^*\vec{A}\right)$ decouples from the linearlized GL equation if we assume that the total vector potential $\vec{A}$ has contributions from the external vector potential only. In other words the order parameter is so weak that we can neglect the screening produced by the superconductor.

The linearized GL equation is a Schrodinger equation with eigenvalue $\epsilon = -\alpha$. There will be an infinite number of such eigenvalues, which correspond to temperatures,
$\epsilon_j = -\alpha_j = \alpha'(1-\frac{T_j}{T_c}) = \alpha'(1-t_j)$. The smallest eigenvalue will correspond to the largest temperature for a solution, $t_j = 1 - \frac{\epsilon_j}{\alpha'}$. This will result in the prediction of the superconductor/normal phase boundary.

B. Calculation of H_{c2} in a Bulk Superconductor

Consider an infinite superconductor in a strong external magnetic field $\vec{H} = H\hat{z}$. This can be represented with the vector potential $\vec{A} = \mu_0 H x\hat{y}$. Put this choice of $\vec{A}$ into the linearized GL equation, divide through by ψ_∞ to form $f = \psi/\psi_\infty$, square the operator and ignore any variation of the order parameter in the direction of the magnetic field (z) to obtain,
$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f + \left(\frac{2\pi}{\Phi_0}\mu_0 H\right)^2 x^2 f + \frac{i4\pi}{\Phi_0}\mu_0 H x\frac{\partial f}{\partial y} = \frac{f}{\xi_{GL}^2}.$
Taking $H = 0$ yields the simple solution $f(x,y) = e^{ik_x x}e^{ik_y y}$ which has uniform magnitude. This suggests the following ansatz for the full equation:
$f(x,y) = g(x)e^{iky}$, where k will be determined later.

Substituting this yields the following second order differential equation for $g(x)$,
$\frac{-\hbar^2}{2m^*}g'' + \frac{1}{2}k_s(x-x_0)^2 g = \frac{\hbar^2}{2m^*}\frac{g}{\xi_{GL}^2} = \epsilon g = -\alpha g,$
where the "spring constant" is $k_s \equiv \frac{(e^*\mu_0 H)^2}{m^*}$, and $x_0 \equiv \frac{\Phi_0 k}{2\pi\mu_0 H}$.
This is the Schrodinger equation for a one-dimensional harmonic oscillator centered on $x = x_0$ with eigenvalues $\epsilon_n = (n+\frac{1}{2})\hbar\omega$ $(n = 0, 1, 2, ...)$ with $\omega = \sqrt{\frac{k_s}{m*}}$. The lowest eigenvalue ($n = 0$) corresponds to the highest t and represents the phase boundary, $\frac{\hbar^2}{2m^*}\frac{1}{\xi_{GL}^2} = (n+\frac{1}{2})\hbar\frac{e^*\mu_0 H}{m^*}$. In this case we are finding a phase boundary in magnetic field where superconductivity first nucleates. The lowest energy eigenvalue of the Schrodinger equation corresponds to the highest magnetic field that supports a non-zero order parameter.
Solving for the magnetic field at the phase boundary (for a given $T < T_c$) yields,
$H_{c2}(T) = \frac{\Phi_0}{2\pi\mu_0}\frac{1}{\xi_{GL}^2}.$

$H_{c2}(T)$ represents the highest field at which superconductivity can nucleate for a given $T < T_c$. Note that as $T \to T_c$, ξ_{GL} diverges and H_{c2} goes to zero. From the temperature dependence of ξ_{GL} we see that near T_c one has $H_{c2}(T) \sim 1 - t$. Thus from the linear slope of $H_{c2}(T)$ at T_c one can deduce the extrapolated zero-temperature GL coherence length. This method is very commonly used by experimentalists to determine the coherence length of superconductors.

One can re-write $H_{c2}(T)$ in various forms using other GL quantities as follows:

$H_{c2} = \frac{4\pi\mu_0}{\Phi_0}\lambda_{eff}^2 H_c^2$,

$H_{c2} = \sqrt{2}\frac{\lambda_{eff}}{\xi_{GL}} H_c = \sqrt{2}\kappa H_c$.
This last expression shows that $H_{c2} > H_c$ for type-II superconductors ($\kappa > 1/\sqrt{2}$), and $H_{c2} < H_c$ for type-I superconductors ($\kappa < 1/\sqrt{2}$). As an example, consider the cuprate superconductor YBCO which has $\mu_0 H_c(0) \sim 1T$ and $\kappa \sim 60$. It's upper critical field $\mu_0 H_{c2}(0) \sim 85T$.

Imagine a superconductor in a large magnetic field at $T < T_c$ such that it is a normal conductor. The magnetic field is now reduced and we examine the evolution of the GL order parameter. For a type-II superconductor the GL order parameter is zero until we reach $H = H_{c2}$ and then it increases continuously as the field is reduced (characteristic of a second order phase transition). This process is reversible and non-hysteretic. Nothing special happens as H is reduced through H_c.
For a type-I superconductor one goes through $H = H_c$ and the order parameter remains zero. Only when the field is reduced to the lower value of $H = H_{c2}$ does superconductivity nucleate and the order parameter suddenly jumps up to its equilibrium value at that temperature and field. This super-cooling process means the superconductor is out of equilibrium. If the field is now increased, the order parameter remains large until the thermodynamic critical field H_c is reached, at which point the order parameter is discontinuously reduced to zero in a first-order transition. The entire cycle is an open hysterisis loop, characteristic of first-order phase transitions.

II. ORDER PARAMETER SOLUTION

We found the first nucleation field for superconductivity in a strong magnetic field for $T < T_c$ as $H_{c2} = \sqrt{2}\kappa H_c$.
We took solutions to the linearized GL equation of the form,
$f(x, y) = g(x)e^{iky}$.
H_{c2} is the first non-trivial solution to the 1D differential equation:
$\frac{-\hbar^2}{2m^*}g'' + \frac{1}{2}k_s(x - x_0)^2 g = \frac{\hbar^2}{2m^*}\frac{g}{\xi_{GL}^2} = \epsilon g = -\alpha g$,
where the "spring constant" is $k_s \equiv \frac{(e^*\mu_0 H)^2}{m^*}$, and $x_0 \equiv \frac{\Phi_0 k}{2\pi\mu_0 H}$.

The eigenfunction for $g(x)$ is the ground state of the harmonic oscillator, with full solutions of the form,
$f(x, y; k) = e^{iky}e^{-(x-x_k)^2/2\xi_{GL}^2}$, with $x_k \equiv \frac{\Phi_0 k}{2\pi\mu_0 H}$.
This represents an infinite number of degenerate solutions, labeled by the parameter k.

We expect a set of solutions that are periodic in space. This can be accomplished by making k an integer multiple of a basic wavenumber q as $k = nq$, with $n = 0, \pm1, \pm2, \ldots$. Now the centers of the Gaussians are also periodic in space with $x_n = \frac{\Phi_0 nq}{2\pi\mu_0 H}$.
The y-solution is periodic with period $\Delta y = 2\pi/q$ and the x-solution has period $\Delta x = \frac{\Phi_0 q}{2\pi\mu_0 H}$. The area of a unit cell is therefore $\Delta x\Delta y = \frac{\Phi_0}{\mu_0 H}$, showing that exactly one flux quantum is confined in each unit cell. This is the ultimate limit imposed by quantum mechanics when the energy per unit area of an S/N interface is negative. The vortex lattice is the result of the proliferation of negative energy interfaces, arrested only by fluxoid quantization.

III. VORTEX LATTICE SOLUTIONS

The general solution for $\psi(x,y)$ comes from a linear superposition of all of the above solutions (recall that we linearized the GL equation, so linear superposition now holds):
$\psi(x,y) = \sum_n C_n e^{inqy} e^{-(x-x_n)^2/2\xi_{GL}^2}$. If C_n is periodic in n then ψ is also periodic in space.
The two common solutions are the square lattice (obtained when $C_n = C_0$ for all n), and the triangular lattice (obtained when $C_1 = iC_0$ and $C_{n+2} = C_n$ for all n). The true minimum energy solution is found from the full nonlinear GL theory by minimizing the free-energy difference,
$\langle f_s - f_n \rangle = -\frac{\alpha^2}{2\beta}\frac{1}{\beta_A}$, with
$\beta_A \equiv \frac{\langle |\psi|^4 \rangle}{\langle |\psi|^2 \rangle^2}$, as derived by Abrikosov. Hence we seek the minimum value of β_A. A uniform solution has $\beta_A = 1$. One finds that the square lattice has $\beta_A = 1.18$ while the triangular lattice has $\beta_A = 1.16$, just slightly lower. The class web site shows these solutions for $\psi(x,y)$ as well as many experimental techniques to image the vortex lattice. Some techniques (STM) measure the local density of states at the Fermi energy, which is enhanced in the vortex core due to the suppressed order parameter. Bitter decoration images the magnetic field concentration near the vortex cores. Lorentz microscopy magneto-optic imaging, SQUID and magnetic force microscopy methods all image the magnetic field profiles.

Note that we have assumed the superconductor does no screening, hence to this first approximation the magnetic field is homogeneous in the superconductor. One can see that the length scale λ_{eff} plays no role in the solution for $\psi(x,y)$, which is a consequence of ignoring the screening.